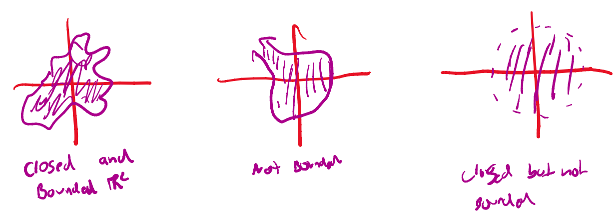
Gradient: direction of maximal increase

Critical point: a point P in the domain of f where either $\nabla f(P) = 0$ or $\nabla f(P)$ does not exist

Fermat's Externa Theorem: If f has a local extreme value at P, then P is a critical point of f

Extreme Value Theorem: If f is defined on a closed and bounded subset $U \subseteq \mathbb{R}^n$, then f obtains its global extrema on U

closed and bounded: A set is closed and bounded iff it is a union of finitely many closed and bounded intervals in $\mathbb{R}^1$



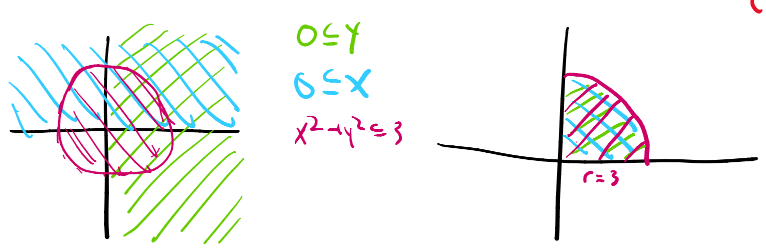
This suggests a method for optimizing global values on a closed and bounded subset

Alg (compact set method) let f be a function defined on a closed and bounded subset U. To compute global extrema of f on U,

1) compute critical points of f on U
2) compute the max and min of f on the critical points
3) optimize f along the boundary of U

The max/min is global extrema values of f on U

Example: find the global extrema of $f(x,y) = xy^2$ on $U = \{(x,y),\ 0 \le x,\ 0 \le y\ \ x^2+y^2 \le 3\}$



Step 1: compute critical points in U

$\nabla f = (y^2, 2xy)$ so $\nabla f = 0$ iff $(y^2, 2xy) = 0$

$y^2 = 0$ iff $y = 0$

$2xy = 0$ iff $x=0$ or $y=0$

So, true iff $y=0$

Note: in this example the boundary points all intersect with U

therefore, we will [illegible]

Step 3: Analyze the boundaries

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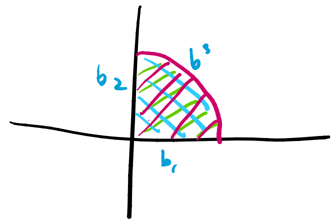


Since $r=3$, b_1 and b_2 will be $\sqrt{3}$

Parametize

$b_1(t) = (t, 0)$ on $0 \le t \le \sqrt{3}$

$b_2(t) = (0, t)$ on $0 \le t \le \sqrt{3}$

$b_3(t) = (\sqrt{3}\cos(t), \sqrt{3}\sin(t))$ on $0 \le t \le \frac{\pi}{2}$

4. Optimization

for b_1: $f(b_1(t)) = f(t, 0) = t \cdot 0^2 = 0$

for b_2: $f(b_2(t)) = f(0, t) = 0 \cdot t^2 = 0$

for b_3: $f(b_3(t)) = f(\sqrt{3}\cos(t), \sqrt{3}\sin(t)) = (\sqrt{3}\cos(t))(\sqrt{3}\sin(t))^2 = 3\sqrt{3}\cos(t)\sin^2(t)$

So, $g'(t) = 3\sqrt{3}\,(-\sin(t)\sin^2 t + \cos(t) \cdot 2\sin(t)\cos(t)) = 3\sqrt{3}\sin(t)(2\cos^2(t) - \sin^2(t))$

$g'(t) = 0$ iff $\sin(t) = 0$ or $2\cos^2 t - \sin^2(t) = 0$

$2\cos^2 t = \sin^2(t)$ · cannot be 0 simultaneously

$\tan^2(t) = 2$

$\tan(t) = \pm\sqrt{2}$

iff $t = 4A$ or $t = \operatorname{atan}(\sqrt{2})$ or $t = \operatorname{atan}(-\sqrt{2})$ less than 0, reject

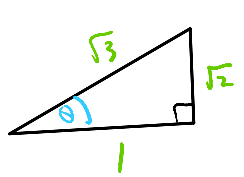
$t = 0$ or $t = \operatorname{atan}\sqrt{2}$

∴ Test $g(t)$

$g(0) = 3\sqrt{3}\cos(0)\sin(0) = 0$

$g\left(\frac{\pi}{2}\right) = 3\sqrt{3}\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) = 0$

$g(\operatorname{atan}\sqrt{2}) = 3\sqrt{3}\cos(\operatorname{atan}\sqrt{2})\sin(\operatorname{atan}\sqrt{2}) = > 0$



$\theta = \operatorname{atan}(\sqrt{2})$

$\theta = \frac{\sqrt{2}}{1}$

$\sin(\theta) = \frac{\sqrt{2}}{\sqrt{3}}$

$\cos\theta = \frac{1}{\sqrt{3}}$

$3\sqrt{3}\left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2 = 2$

$\boxed{\text{The abs max of } u = 2 \quad \text{The abs min of } u = 0}$

First Derivative Test

let f be diff at critical point $\vec{p}$

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let f be diff at critical point $\vec{p}$

① if $D_{\vec{v}} f(\vec{p} + \varepsilon\vec{v}) > 0$ for all suf. small $\varepsilon > 0$ and all unit vectors $\vec{v}$, then f has a local min at $\vec{p}$.

② if $D_{\vec{v}} f(\vec{p} + \varepsilon\vec{v}) < 0$ for all suf. small $\varepsilon > 0$ and all unit vectors $\vec{v}$, then f has a local max at $\vec{p}$

- This is too hard to use on a problem

Second derivitave Test

Warning! failure is possible

$$D = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = f_{xx} \cdot f_{yy} - f_{xy} \cdot f_{yx} = f_{xx} \cdot f_{yy} - f_{xy}^{2}$$

Definition

This only works as stated for $f(x,y)$

1) If $f_{xx}(\vec{p}) > 0$ and $D(\vec{p}) \cdot f_{yy}(\vec{p}) - f_{xy}(\vec{p})^2 > 0$ than $\vec{p}$ is a point with local min of f

2) If $f_{xx}(\vec{p}) < 0$ and $D\,\vec{p} \cdot f_{yy}(\vec{p}) - f_{xy}(\vec{p})^2 > 0$ than $\vec{p}$ is a point with local max of f

3) If $D\,\vec{p} \cdot f_{yy}(\vec{p}) - f_{xy}(\vec{p})^2 = 0$ than $\vec{p}$ is a sattelite point